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Engineering Notes

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Asymptotic Theory and Limiting Cases for Spinning Spacecraft Subject to Constant Forces

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Introduction

In 1965, Armstrong [1] developed analytical solutions for thrusting, spinning-up, axisymmetric rigid bodies and considered some geometric cases for high spin rates. He introduced compact results for translational and rotational motion of a rigid body in space in the presence of body-fixed forces and moments. Although his results are important, they are not complete and, unfortunately, they have not been presented in a conference or published in an archived journal. In his doctoral thesis, Ayoubi [2] verified and completed Armstrong's asymptotic theory for thrusting and for spinning-up rigid bodies.

In other work, Longuski and Kia [3], Beck and Longuski [4], and Javorsek and Longuski [5] presented asymptotic limits for some special cases for thrusting and spinning-up spacecraft when the body-fixed forces and moments are constant and when the body is axisymmetric or nearly axisymmetric. In this paper, we use the results introduced by Longuski et al. [6] for angular velocity, Eulerian angles, and transverse and axial displacements and examine those solutions for limiting cases of the spinning, thrusting rigid body in which the spin rate remains nearly constant.

Angular Velocity

Let us consider a nearly axisymmetric (or axisymmetric) spin-stabilized rigid body in an inertial frame, as shown in Fig. 1. If the body is subjected to constant body-fixed forces and constant transverse moments and the two Eulerian angles ϕ_x and ϕ_y are small enough, then the spin rate about the near-axisymmetric axis is approximated by a constant, ω_{z0}, and the scaled angular velocity can be written [6] as

$$\Omega(t) = \Omega_0 e^{ik_{xy}\omega_{z0}t} + \frac{iF}{k_{xy}\omega_{z0}}(1 - e^{ik_{xy}\omega_{z0}t}), \qquad \Omega_0 \triangleq \Omega(t_0) \quad (1)$$

where

$$\Omega \triangleq \Omega_x + i\Omega_y, \qquad \Omega_x \triangleq \omega_x\sqrt{k_y}, \qquad \Omega_y \triangleq \omega_y\sqrt{k_x} \quad (2)$$

$$k_x \triangleq \frac{I_z - I_y}{I_x}, \qquad k_y \triangleq \frac{I_z - I_x}{I_y}, \qquad k_{xy} \triangleq \sqrt{k_x k_y} \quad (3)$$

and where

$$F \triangleq F_x + iF_y, \qquad F_x \triangleq \frac{M_x\sqrt{k_y}}{I_x}, \qquad F_y \triangleq \frac{M_y\sqrt{k_x}}{I_y} \quad (4)$$

Without loss of generality, we assume $I_z > I_y > I_x$.

After some algebra, it can be shown [from Eq. (1)] that when $k_y \leq k_x$, the upper bound of the angular velocity is given by

$$|\omega| = \sqrt{\omega_x^2 + \omega_y^2} \leq |\omega_0| + \frac{2|F|}{k_y\omega_{z0}\sqrt{k_x}} \quad (5)$$

For the axisymmetric case, we can show that Eq. (5) reduces to

$$|\omega| \leq |\omega_0| + \frac{2(k+1)}{|k|}\frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}} \quad (6)$$

where

$$k = k_x = k_y = \frac{I_z}{I_x} - 1 \quad (7)$$

In the following, we consider some special geometric cases (i.e., a sphere, a thin rod, and a flat disk) to find the upper limit bound of the scaled angular velocity.

Sphere ($k \to 0$)

In the case of a sphere, we have $I_x = I_y = I_t$ and $I_z \to I_t$; thus, in Eq. (7), $k \to 0$. We see that the right-hand side of Eq. (6) has a zero divisor, and so

$$|\omega| \to \infty \quad (8)$$

We are not surprised by the result of Eq. (8), because in the case of spherical symmetry, Euler's equations of motion simplify to

$$\dot{\omega}_x = \frac{M_x}{I_t}, \qquad \dot{\omega}_y = \frac{M_y}{I_t}, \qquad \dot{\omega}_z = 0 \quad (9)$$

and so the transverse angular velocity components have secular terms proportional to time t. In addition, we expect the Euler angles to grow with t^2, as we can surmise from the 3-1-2 kinematic equations [2].

Thin Rod ($k \to -1$)

For a thin rod, we know that $I_x = I_y = I_t$ and $I_z \to 0$; thus, in Eq. (7), $k \to -1$. In this case, the limit of Eq. (6) will be

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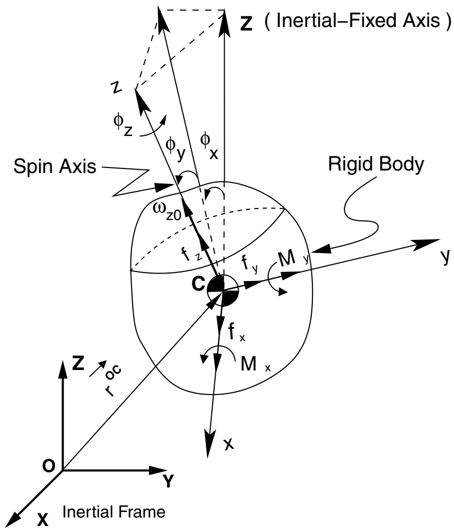

Fig. 1 A model for a spinning rigid body in an inertial frame. Body-fixed forces (f_x, f_y, f_z) and transverse body-fixed moments (M_x, M_y) are constant. The spin rate about the near-axisymmetric axis is approximated by a constant, ω_{z0}.

$$|\omega| \leq |\omega_0| \tag{10}$$

Flat Disk ($k \to +1$)

For a plane cylinder, we have $I_x = I_y = I_t$ and $I_z = 2I_t$. In the limiting case for a flat disk, $k \to +1$. It can be shown [2] that the limit of Eq. (6) is

$$|\omega| \leq |\omega_0| + \frac{4\sqrt{M_x^2 + M_y^2}}{I_z \omega_{z0}} \tag{11}$$

Eulerian Angles

We use the Eulerian angle solution, which is given by Longuski et al. [6] as

$$\phi(t) = \phi_0 e^{-i\omega_{z0}t} + e^{-i\omega_{z0}t} I_\phi(t), \qquad \phi_0 \triangleq \phi(0) \tag{12}$$

where I_ϕ is

$$I_\phi(t) = -\frac{ik_1\Omega_0}{\mu\omega_{z0}}(e^{i\mu\omega_{z0}t} - 1) + \frac{k_1 F}{k\omega_{z0}^2}\left[(e^{i\omega_{z0}t} - 1) - \frac{(e^{i\mu\omega_{z0}t} - 1)}{\mu}\right] - \frac{ik_2\bar{\Omega}_0}{\kappa\omega_{z0}}(e^{i\kappa\omega_{z0}t} - 1) - \frac{k_2\bar{F}}{k\omega_{z0}^2}\left[(e^{i\omega_{z0}t} - 1) - \frac{(e^{i\kappa\omega_{z0}t} - 1)}{\kappa}\right] \tag{13}$$

and where an overbar denotes a complex conjugate. The parameters μ and κ are defined as follows:

$$\mu \triangleq \lambda + \rho = \lambda(1 + k_{xy}), \qquad \kappa \triangleq \lambda - \rho = \lambda(1 - k_{xy}) \tag{14}$$

and

$$k_1 \triangleq \frac{\sqrt{k_x} + \sqrt{k_y}}{2k}, \qquad k_2 \triangleq \frac{\sqrt{k_x} - \sqrt{k_y}}{2k} \tag{15}$$

From Eq. (12), we can obtain [2]

$$|\phi| \leq |\phi_0| + \frac{2|\Omega_0|}{\omega_{z0}}\left[\left|\frac{k_1}{\mu}\right| + \left|\frac{k_2}{\kappa}\right|\right] + \frac{2|F|}{|k|\omega_{z0}^2}(|k_1| + |k_2|) + \frac{2|F|}{|k|\omega_{z0}^2}\left[\left|\frac{k_1}{\mu}\right| + \left|\frac{k_2}{\kappa}\right|\right] \tag{16}$$

For axisymmetric cases, we note that

$$k_1 = \frac{1}{\sqrt{|k|}}, \qquad k_2 = 0 \tag{17}$$

Using Eq. (7), we can show that Eq. (16) reduces to

$$|\phi| \leq |\phi_0| + \frac{2|\Omega_0|}{\omega_{z0}\sqrt{|k|}} + \frac{2\sqrt{M_x^2 + M_y^2}}{I_z \omega_{z0}^2}\left(\frac{k+2}{|k|}\right) \tag{18}$$

In the following, we consider limiting cases for the Eulerian angles for a sphere ($k \to 0$), a thin rod ($k \to -1$), and a flat disk ($k \to +1$).

Sphere ($k \to 0$)

In this case, the second term on the right-hand side of Eq. (18) grows without bound; thus, we have

$$|\phi| \to \infty \tag{19}$$

Because Eq. (18) is not valid for large values of ϕ, Eq. (19) is meaningful only in the sense that $|\phi|$ initially tends to grow without bound.

Thin rod ($k \to -1$)

The limit of Eq. (18) when $k \to -1$ is

$$|\phi| \leq |\phi_0| + \frac{2|\Omega_0|}{\omega_{z0}} + \frac{2\sqrt{M_x^2 + M_y^2}}{I_z \omega_{z0}^2} \tag{20}$$

Flat disk ($k \to +1$)

The limit of Eq. (18) when $k \to +1$ is

$$|\phi| \leq |\phi_0| + \frac{2|\Omega_0|}{\omega_{z0}} + \frac{6\sqrt{M_x^2 + M_y^2}}{I_z \omega_{z0}^2} \tag{21}$$

We notice for the special case when $|\Omega_0| = |\phi_0| = 0$ that $|\phi_{\text{disk}}| = 3|\phi_{\text{rod}}|$.

Transverse and Axial Velocities, Case I ($f_x = f_y = 0, f_z, M_x$, and $M_y \neq 0$)

By using the results of Longuski et al. [6], the magnitude of the axial velocity can be written as

$$\Delta v_{Z\,\text{sec}} = \frac{f_z}{m} t \tag{22}$$

and the transverse velocity is bounded by

$$|\Delta v_{XY}| \leq \frac{|f_z| t}{m}\left[|\phi_0| + \left|\frac{2k_1 F}{k\omega_{z0}^3}\right| + \left|\frac{2k_1\Omega_0}{\mu^2\omega_{z0}^2}\right| + \left|\frac{2k_1 F}{k\mu^2\omega_{z0}^3}\right| + \left|\frac{k_1\Omega_0}{\mu\omega_{z0}}\right| + \left|\frac{k_1 F}{\mu\omega_{z0}^2}\right| + \left|\frac{2k_2 F}{k\omega_{z0}^3}\right| + \left|\frac{2k_2\Omega_0}{\kappa^2\omega_{z0}^2}\right| + \left|\frac{2k_2 F}{k\kappa^2\omega_{z0}^3}\right| + \left|\frac{k_2\Omega_0}{\kappa\omega_{z0}}\right| + \left|\frac{k_2 F}{\kappa\omega_{z0}^2}\right|\right] + \frac{2\sqrt{f_x^2 + f_y^2}}{m\omega_{z0}} \tag{23}$$

Here, we see that the first term (which is a secular term), appears in Eq. (23), indicating that a component of the axial velocity is projected into the X–Y inertial plane. If the axial acceleration is set to zero (i.e., $f_z = 0$), then Eq. (23) provides a constant transverse velocity.

For case I when $f_x = f_y = 0$, Eq. (23) reduces to

$$|\Delta v_{XY}| \le \frac{|f_z|t}{m}\left\{|\phi_0| + |k_1|\left(\left|\frac{2F}{k\omega_{z0}^3}\right| + \frac{2|\Omega_0|}{\mu^2\omega_{z0}^2} + \left|\frac{2F}{k\mu^2\omega_{z0}^3}\right| + \left|\frac{\Omega_0}{\mu\omega_{z0}}\right| + \left|\frac{F}{\mu\omega_{z0}^2}\right|\right) + |k_2|\left(\left|\frac{2F}{k\omega_{z0}^3}\right| + \frac{2|\Omega_0|}{\kappa^2\omega_{z0}^2} + \left|\frac{2F}{k\kappa^2\omega_{z0}^3}\right| + \left|\frac{\Omega_0}{\kappa\omega_{z0}}\right| + \left|\frac{F}{\kappa\omega_{z0}^2}\right|\right)\right\} \tag{24}$$

For axisymmetric cases, Eq. (24) simplifies to

$$|\Delta v_{XY}| \le \frac{|f_z|t}{m}\left\{|\phi_0| + \frac{1}{(k+1)\sqrt{|k|}}\left[1 + \frac{2}{(k+1)\omega_{z0}}\right]\frac{|\Omega_0|}{\omega_{z0}} + \frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}^2}\left[1 + \frac{2(k+1)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right]\right\} \tag{25}$$

When $|\Omega_0| = |\phi_0| = 0$, Eqs. (24) and (25) reduce to

$$|\Delta v_{XY}| \le \frac{2|Ff_z|t}{|mk\omega_{z0}^3|}\left[|k_1|\left|1 + \frac{1}{\mu^2} + \frac{\omega_{z0}}{2\mu}\right| + |k_2|\left|1 + \frac{1}{\kappa^2} + \frac{\omega_{z0}}{2\kappa}\right|\right] \tag{26}$$

$$|\Delta v_{XY}| \le \frac{|f_z|t}{m}\frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}^2}\left[1 + \frac{2(k+1)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right] \tag{27}$$

Sphere ($k \to 0$)

In the case of a sphere, the limit of Eq. (25) when $k \to 0$ is

$$|\Delta v_{XY}| \to \infty \tag{28}$$

We note that the 2nd, 4th, 5th, and 6th terms on the right-hand side of Eq. (25) have zero divisors, indicating that the transverse velocity increases without limit because a component of the transverse force remains fixed in the inertial frame.

Thin Rod ($k \to -1$)

In a similar manner to that of a sphere, we can show that in the case of a thin rod, we have

$$|\Delta v_{XY}| \to \infty \tag{29}$$

The 2nd, 3rd, and 6th terms on the right-hand side of Eq. (25) grow without bound.

Flat Disk ($k \to +1$)

The limit of Eq. (25) when $k \to +1$ is given by

$$|\Delta v_{XY}| \le |\frac{|f_z|t}{m}\left[|\phi_0| + \frac{|\Omega_0|}{2\omega_{z0}}\left(1 + \frac{1}{\omega_{z0}}\right) + \frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}^2}\left(5 + \frac{1}{\omega_{z0}}\right)\right] \tag{30}$$

which also exhibits secular behavior.

Transverse and Axial Velocities, Case II (f_x, f_y, f_z, M_x, and $M_y \neq 0$)

We introduce the magnitude of the transverse velocity [Eq. (23)] when all the body-fixed forces and moments are present (except the axial moment, which is zero). It can easily be seen that the results of the case I are applicable in case II. It is interesting to note that when $f_x, f_y \neq 0$, and $f_z = 0$, Eq. (23) reduces to

$$|\Delta v_{XY}| = \frac{2\sqrt{f_x^2 + f_y^2}}{m\omega_{z0}} \tag{31}$$

which was introduced by Beck and Longuski [4].

Next, we consider the axial velocity. Using the results of [6] and retaining all secular terms, we can get the following equation for $\Delta v_{z\,\mathrm{sec}}$:

$$\Delta v_{Z\,\mathrm{sec}} = \frac{f_z t}{m} - \frac{1}{m}\Im\left[\bar{f}\int_0^t \phi(\tau)\,\mathrm{d}\tau\right]_{\mathrm{sec}} = \frac{f_z t}{m} - \frac{1}{m}\Im\left[\bar{f}\left(\frac{k_1 F t}{k\omega_{z0}^2} - \frac{k_2 \bar{F} t}{k\omega_{z0}^2}\right)\right] \tag{32}$$

where Im[·] means the imaginary part.

After some algebra, Eq. (32) can be simplified to

$$\Delta v_{Z\,\mathrm{sec}} = \frac{f_z t}{m} + \frac{1}{m\omega_{z0}^2}\left(\frac{f_y M_x}{k_x I_x} - \frac{f_x M_y}{k_y I_y}\right)t \tag{33}$$

For the axisymmetric case, Eq. (33) reduces to

$$\Delta v_{Z\,\mathrm{sec}} = \frac{f_z t}{m}\left[1 + \frac{(k+1)}{k\omega_{z0}^2}\left(\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right)\right] \tag{34}$$

Now we consider the following special cases.

Sphere ($k \to 0$)

For a sphere, it can be seen that the second term on the right-hand side of Eq. (34) has a zero divisor when $k \to 0$; thus,

$$\Delta v_{Z\,\mathrm{sec}} \to \infty \tag{35}$$

Thin Rod ($k \to -1$)

For a thin rod, we can show that limit of Eq. (34) is

$$\Delta v_{Z\,\mathrm{sec}} = \frac{f_z t}{m} \tag{36}$$

When $t \to \infty$, then $\Delta v_{Z\,\mathrm{sec}} \to \infty$.

Flat Disk ($k \to +1$)

In the case of a flat disk, the limit of Eq. (34) is

$$\Delta v_{Z\,\mathrm{sec}} = \frac{f_z t}{m}\left[1 + \frac{2}{\omega_{z0}^2}\left(\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right)\right] \tag{37}$$

We notice that when $t \to \infty$, $\Delta v_{Z\,\mathrm{sec}} \to \infty$. From the transverse and axial velocity solutions, we can proceed to find the velocity bias angle (the angle between the velocity vector and the inertial Z frame) for cases I and II in the following section.

Velocity Bias Angle, Case I ($f_x = f_y = 0, f_z, M_x$, and $M_y \neq 0$)

The velocity bias angle ρ is defined as the angle between the velocity vector and the inertial Z axis (i.e., the desired velocity direction). With the assumption that the Euler angles ϕ_x and ϕ_y are small, we can write

$$|\rho_{\mathrm{sec}}| \simeq |\tan(\rho_{\mathrm{sec}})| = \left|\frac{\Delta v_{XY}}{\Delta v_{Z\,\mathrm{sec}}}\right| \tag{38}$$

By dividing Eq. (24) by Eq. (22), we get

$$
\begin{aligned}
|\rho_{\text{sec}}| \le |\phi_0| + |k_1|\Bigg(&\left|\frac{2F}{k\omega_{z0}^3}\right| + \frac{2|\Omega_0|}{\mu^2\omega_{z0}^2} \\
&+ \left|\frac{2F}{k\mu^2\omega_{z0}^3}\right| + \left|\frac{\Omega_0}{\mu\omega_{z0}}\right| + \left|\frac{F}{\mu\omega_{z0}^2}\right|\Bigg) \\
&+ |k_2|\left(\left|\frac{2F}{k\omega_{z0}^3}\right| + \frac{2|\Omega_0|}{\kappa^2\omega_{z0}^2} + \left|\frac{2F}{k\kappa^2\omega_{z0}^3}\right| + \left|\frac{\Omega_0}{\kappa\omega_{z0}}\right| + \left|\frac{F}{\kappa\omega_{z0}^2}\right|\right)
\end{aligned} \tag{39}
$$

It can be shown [2] that for the axisymmetric case, Eq. (39) reduces to

$$
\begin{aligned}
|\rho_{\text{sec}}| \le |\phi_0| &+ \frac{1}{(k+1)\sqrt{|k|}}\left[1 + \frac{2}{(k+1)\omega_{z0}}\right]\frac{|\Omega_0|}{\omega_{z0}} \\
&+ \frac{\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left[1 + \frac{2(k+1)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right]
\end{aligned} \tag{40}
$$

If the initial conditions are set to zero (i.e., $|\phi_0| = |\Omega_0| = 0$), Eq. (40) simplifies to

$$
|\rho_{\text{sec}}| \le \frac{\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left[1 + \frac{2(k+1)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right] \tag{41}
$$

which was introduced by Longuski [7]. In the following, we consider the limit of Eq. (40) for a sphere, a thin rod, and a flat disk. We note that when $k \to 0$ or $k \to -1$, the second term on the right-hand side of Eq. (40) grows without bound.

Sphere ($k \to 0$)

It can be shown that the limit of Eq. (40) when $k \to 0$ is

$$
|\rho_{\text{sec}}| \to \infty \tag{42}
$$

Thin Rod ($k \to -1$)

Similar to the sphere, we can show that the limit of Eq. (40) grows without bound when $k \to -1$; thus,

$$
|\rho_{\text{sec}}| \to \infty \tag{43}
$$

The only meaning we ascribe to Eqs. (42) and (43) is that ρ_{sec} tends to grow beyond the limit of the validity of Eq. (40).

Flat Disk ($k \to +1$)

For the flat disk, we have the bound

$$
|\rho_{\text{sec}}| \le |\phi_0| + \frac{|\Omega_0|}{2\omega_{z0}}\left(1 + \frac{1}{\omega_{z0}}\right) + \frac{\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left(5 + \frac{1}{\omega_{z0}}\right) \tag{44}
$$

Velocity Bias Angle, Case II (f_x, f_y, f_z, M_x, and $M_y \neq 0$)

Similar to case I, the velocity bias angle can be derived by dividing Eq. (23) by Eq. (33). For the axisymmetric case, we divide Δv_{XY} [Eq. (25)] by $\Delta v_{Z\,\text{sec}}$ [Eq. (34)] and set the initial conditions to zero (i.e., $|\phi_0| = |\Omega_0| = 0$). We get

$$
\begin{aligned}
|\rho_{\text{sec}}| &\simeq \left|\frac{\Delta v_{XY\,\text{sec}}}{\Delta v_{Z\,\text{sec}}}\right| \\
&\le \left\{\frac{\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left[1 + \frac{2(k+1)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right]\right\} \\
&\quad \times \left[\left|1 + \frac{(k+1)}{k\omega_{z0}^2}\left(\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right)\right|\right]^{-1}
\end{aligned} \tag{45}
$$

Now we determine the limit of Eq. (45) when $k \to 0, -1$, and $+1$ as follows.

Sphere ($k \to 0$)

We can show that in the spherical case, the limit of Eq. (45) goes to

$$
|\rho_{\text{sec}}| \to \left[\frac{2\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left(1 + \frac{1}{\omega_{z0}}\right)\right]\left[\frac{1}{\omega_{z0}^2}\left|\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right|\right]^{-1} \tag{46}
$$

Thin Rod ($k \to -1$)

For the thin rod, we have

$$
|\rho_{\text{sec}}| \to \infty \tag{47}
$$

and so there is no small, finite limit for $|\rho_{\text{sec}}|$.

Flat Disk ($k \to +1$)

For the flat disk, we have secular behavior:

$$
\begin{aligned}
|\rho_{\text{sec}}| \to &\left[\frac{\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left(5 + \frac{1}{\omega_{z0}}\right)\right] \\
&\times \left[\left|1 + \frac{2}{\omega_{z0}^2}\left(\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right)\right|\right]^{-1}
\end{aligned} \tag{48}
$$

In the following sections, we consider the displacement solutions for cases I and II.

Transverse and Axial Displacements, Case I ($f_x = f_y = 0; f_z, M_x$, and $M_y \neq 0$)

To get the secular component of the displacement, we start with the displacement solution given in [6] and retain only those terms that include secular terms; alternatively, we can use our velocity solution [Eqs. (23) and (25)] and integrate those equations. Because the second method is straightforward and easier than the first, we choose the second method to derive the displacement solutions as follows.

Integrating Eq. (23) yields

$$
\begin{aligned}
|\Delta d_{XY}| \le \frac{|f_z|t^2}{2m}\Bigg\{|\phi_0| + |k_1|\Bigg(&\left|\frac{2F}{k\omega_{z0}^3}\right| + \frac{2|\Omega_0|}{\mu^2\omega_{z0}^2} \\
&+ \left|\frac{2F}{k\mu^2\omega_{z0}^3}\right| + \left|\frac{\Omega_0}{\mu\omega_{z0}}\right| + \left|\frac{F}{\mu\omega_{z0}^2}\right|\Bigg) \\
&+ |k_2|\left(\left|\frac{2F}{k\omega_{z0}^3}\right| + \frac{2|\Omega_0|}{\kappa^2\omega_{z0}^2} + \left|\frac{2F}{k\kappa^2\omega_{z0}^3}\right| + \left|\frac{\Omega_0}{\kappa\omega_{z0}}\right| + \left|\frac{F}{\kappa\omega_{z0}^2}\right|\right)\Bigg\}
\end{aligned} \tag{49}
$$

which, for axisymmetric cases, reduces to

$$
\begin{aligned}
|\Delta d_{XY}| \le \frac{|f_z|t^2}{2m}\Bigg\{|\phi_0| &+ \frac{1}{(k+1)\sqrt{|k|}}\left[1 + \frac{2}{(k+1)\omega_{z0}}\right]\frac{|\Omega_0|}{\omega_{z0}} \\
&+ \frac{\sqrt{M_x^2+M_y^2}}{I_z\omega_{z0}^2}\left[1 + \frac{2(1+k)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right]\Bigg\}
\end{aligned} \tag{50}
$$

When $|\Omega_0| = |\phi_0| = 0$, Eqs. (49) and (50) reduce to

$$
|\Delta d_{XY}| \le \frac{2|Ff_z|t^2}{|2mk\omega_{z0}^3|}\left[|k_1|\left|1 + \frac{1}{\mu^2} + \frac{\omega_{z0}}{2\mu}\right| + |k_2|\left|1 + \frac{1}{\kappa^2} + \frac{\omega_{z0}}{2\kappa}\right|\right] \tag{51}
$$

and

$$|\Delta d_{XY}| \leq \frac{|f_z|t^2}{2m} \frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}^2} \left[1 + \frac{2(k+1)}{|k|} + \frac{2}{|k|(k+1)\omega_{z0}}\right] \quad (52)$$

Similarly, the axial displacement can be derived by integrating Eq. (22) as

$$\Delta d_{Z\,\mathrm{sec}} = \frac{f_z t^2}{2m} \quad (53)$$

The following limiting cases are investigated as follows.

Sphere ($k \to 0$)

Because the velocity goes to infinity in the spherical case, then the displacement goes to infinity:

$$|\Delta d_{XY}| \to \infty \quad (54)$$

Thin Rod ($k \to -1$)

Similar to the spherical case, we see that in this case, the displacement goes to infinity:

$$|\Delta d_{XY}| \to \infty \quad (55)$$

Equations (54) and (55) indicate that expression (52) cannot provide a valid approximation for $|\Delta d_{XY}|$ in the case of a thin rod or a sphere.

Flat Disk ($k \to +1$)

For the flat disk, we have

$$|\Delta d_{XY}| \leq \frac{|f_z|t^2}{2m}\left[|\phi_0| + \frac{|\Omega_0|}{2\omega_{z0}}\left(1 + \frac{1}{\omega_{z0}}\right) + \frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}^2}\left(5 + \frac{1}{\omega_{z0}}\right)\right] \quad (56)$$

Transverse and Axial Displacements, Case II (f_x, f_y, f_z, M_x, and $M_y \neq 0$)

Similar to case I, we can find the transverse and axial displacements for case II by integrating Eq. (33) as follows:

$$\Delta d_{Z\,\mathrm{sec}} = \frac{f_z t^2}{2m} + \frac{t^2}{2m\omega_{z0}^2}\left(\frac{f_y M_x}{k_x I_x} - \frac{f_x M_y}{k_y I_y}\right) \quad (57)$$

For the axisymmetric case, we obtain

$$\Delta d_{Z\,\mathrm{sec}} = \frac{f_z t^2}{2m} + \frac{(k+1)}{k}\frac{(f_y M_x - f_x M_y)t^2}{2mI_z\omega_{z0}^2} \quad (58)$$

which can be rewritten in the form

$$\Delta d_{Z\,\mathrm{sec}} = \frac{f_z t^2}{2m}\left[1 + \frac{(k+1)}{k\omega_{z0}^2}\left(\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right)\right] \quad (59)$$

The limit of Eq. (59) when $k \to 0, -1$, and $+1$ are considered, respectively, is as follows:

Sphere ($k \to 0$)

In the case of a sphere, the second term on the right-hand side of Eq. (59) grows without limit; thus, we have

$$\Delta d_{Z\,\mathrm{sec}} \to \infty \quad (60)$$

and so we have no valid approximation for the growth of $|\Delta d_{Z\,\mathrm{sec}}|$.

Thin Rod ($k \to -1$)

For a thin rod, we have

$$\Delta d_{Z\,\mathrm{sec}} = \frac{f_z t^2}{2m} \quad (61)$$

Flat Disk ($k \to +1$)

In the case of the flat disk, we have secular behavior:

$$\Delta d_{Z\,\mathrm{sec}} = \frac{f_z t^2}{2m}\left[1 + \frac{2}{\omega_{z0}^2}\left(\frac{f_y}{f_z}\frac{M_x}{I_z} - \frac{f_x}{f_z}\frac{M_y}{I_z}\right)\right] \quad (62)$$

Conclusions

We use the closed-form approximate analytical solutions in the literature for a spinning, thrusting spacecraft to develop an asymptotic theory and to deduce limiting cases. We assume that the body is subjected to constant body-fixed forces about all three body axes and constant transverse torques, whereas the axial torque is assumed to be zero. We use the analytical solutions to find an upper bound for the magnitude of the angular velocity, Eulerian angles, transverse and axial velocities, and transverse and axial displacements. These bounds are derived for high spin rates and for the geometric limiting cases of a sphere, a thin rod, and a flat disk. These bounds are valid for axisymmetric, nearly axisymmetric, and (under special conditions) asymmetric rigid bodies, as long as two Eulerian angles remain small.

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